Model Order Reduction of Large Circuits Using Balanced Truncation Via the Arnoldi Method

Payam Rabiei, Massoud Pedram
Department of EE - Systems
University of Southern California
Los Angeles, California

Outline

- Introduction
- State Space Form and System Norm
- Model Reduction Based on Balanced Truncation
- Numerical Methods
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Introduction

- In high-frequency range, circuits should be modeled as distributed elements
- Extracted circuits are huge and cannot be simulated without order reduction
- AWE is a method for order reduction based on Pade approximation of the system transfer function
- Improvements of AWE include RICE, PVL and PRIMA (guarantees passivity)

Overview

- Model order reduction has extensively been studied in the control engineering
- Both frequency- and time-domain model reduction techniques have been proposed
- An effective method is based on the balanced realization of the system
  - Guaranteed stability
  - Bound on the error for the reduced system
  - Provably optimal solution

State Space Form

- Any linear, time-invariant circuit can be written in *standard state space form* as :

$$\dot{\vec{x}} = A\vec{x} + B\vec{u}$$

$$\vec{y} = C\vec{x} + D\vec{u}$$

- Given the state space matrix (A,B,C,D), the transfer function of the system is:

$$G(s) = C(sI - A)^{-1}B + D$$

- Without loss of generality, assume D=0

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Singular Value Decomposition

- Any $l \times m$ matrix ***A*** may be factorized into a singular value decomposition $A = U\Sigma V^H$ where the $l \times l$ matrix ***U*** and the $m \times m$ matrix ***V*** are unitary and the $l \times m$ matrix Σ contains a diagonal matrix Σ_1 of real, non-negative singular values σ_i arranged as $[\Sigma_1\ 0]$ if $l < m$, as Σ_1 if $l = m$ and as $[\Sigma_1\ 0]^T$ otherwise.

  Note that $\Sigma_1 = diag\{\sigma_1, \sigma_2, \cdots, \sigma_k\}$; $k = \min(l, m)$ and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k$ the singular values σ_i are positive square roots of the ***k*** largest eigen-values of both AA^H and $A^H A$. Matrices ***U*** and ***V*** are unit eigen-vectors of AA^H and $A^H A$.

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Balancing the System

- The idea is to find a state vector realization of the system that results in equal coupling of energy from the inputs to states and from states to the outputs
- The *Reachability and Observibility Gramians* are measures of such energy couplings:

$$W_r = \int_0^{\infty} e^{A\tau} B B^T e^{A^T \tau} d\tau \qquad W_o = \int_0^{\infty} e^{A^T\tau} C^T C e^{A\tau} d\tau$$

- The Gramians are obtained by solving the Lyapunov equations

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Lyapunov Equations

- The Lyapunov equations can be stated as follows:

$$AW_r + W_r A^T + BB^T = 0$$

$$A^T W_o + W_o A + C^T C = 0$$

- The Hankel singular values and the Hankel norm are then calculated as:

$$\sigma_i(G(s)) = \sqrt{\lambda_i \ (W_r W_o)}, i = 1,2,\cdots,n \qquad \sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_n \ge 0$$

$$\|G(s)\|_H = \max \sigma_i = \sigma_1$$

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Time-Domain View of Hankel Norm

- It can be shown that Hankel norm is also given by:

$$\|G(s)\|_H = \max_{\vec{u}(t)} \frac{\sqrt{\int_0^{\infty} \|\vec{y}(t)\|_2^2}}{\sqrt{\int_{-\infty}^{0} \|\vec{u}(t)\|_2^2}}$$

- Hankel norm can be interpreted as a kind of induced norm from past inputs to future outputs

Balanced Realization (BR)

- By applying a transformation T to the system, we can change W_r and W_o as follows:

$$\hat{W}_r = T^{-1} W_r T^{-T} \qquad \hat{W}_o = T^T W_o T$$

- It can be shown that for any system, there is a transformation which makes

$$\hat{W}_r = \hat{W}_o$$

- Using such a transformation, the new system is called a *balanced realization*

Reduced System

- To reduce order of the system, we simply ignore states with small Hankel singular values:

$$A=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}, B=\begin{bmatrix}B_1\\ B_2\end{bmatrix}, C=[C_1 \quad C_2] \quad \Sigma=\begin{bmatrix}\Sigma_1 & 0\\ 0 & \Sigma_2\end{bmatrix}$$

- Original System $G(s)=C(sI-A)^{-1}B$
- Reduced System $G_a^k(s)=C_1(sI-A_{11})^{-1}B_1$

H_∞ System Norm

- H_∞ norm of the system is defined as:

$$\|G(s)\|_\infty = \max_\omega \sigma_1(G(j\omega))$$

- Note that given any matrix A, σ_i is the defined as: $\sigma_i(A)=\sqrt{\lambda_i(A^H A)}=\sqrt{\lambda_i(AA^H)}$
- σ_1 is the largest σ_i
- It can be shown that: $\|G(s)\|_\infty = \max_{\vec{u}(t)\neq 0} \frac{\|\vec{y}(t)\|_2}{\|\vec{u}(t)\|_2}$
- So when $\|G_1(s)-G_2(s)\|_\infty \approx 0$, the two systems are almost identical

Main Theorems

- Let $G(s)$ denote a stable rational transfer function of degree n with Hankel singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$. Let $G_a^k(s)$ denote the order k reduction of this transfer function as defined previously. We have:

$$\|G(s) - G_a^k(s)\|_\infty \leq 2(\sigma_{k+1} + \sigma_{k+2} + \cdots + \sigma_n)$$

- The minimum error for approximating $G(s)$ with an arbitrary transfer function $H(s)$ of degree $r < n$ is given by:

$$\|G(s) - H(s)\|_\infty \geq \sigma_{k+1}$$

Numerical Methods

- **We can directly obtain the reduced order system without calculating the BR**
- **Procedure (Safanov'89):**
  - Compute matrices $V_{L,k}$ and $V_{R,k}$ whose columns form bases for the right and left eigen-spaces of $W_r W_o$ associated with the big eigen-values $\sigma_1^2, \cdots, \sigma_k^2$
  - Set $E = V_{L,k}{}^T V_{R,k}$
  - Compute singular value decomposition $U_E \Sigma_k V_E{}^T = E$
  - Set

$$S_L = V_{L,k} U_E \Sigma^{-1/2} \in \Re^{n \times k}$$

$$S_R = V_{R,k} V_E \Sigma^{-1/2} \in \Re^{n \times k}$$

  - The reduced order system is given by:

$$\hat{A} = S_L{}^T A S_R \qquad \hat{B} = S_L{}^T B \qquad \hat{C} = C S_R \qquad \hat{D} = D$$

Numerical Methods

- In Safanov's algorithm, we need W_r and W_o and then the Schur decomposition of $W_r W_o$ to obtain $V_{L,k}$ and $V_{R,k}$
- Large Lyapunov equations can be solved directly using Krylov-subspace methods (based on the Arnoldi algorithm) as shown in [Saad'89]
- For the decomposition of $W_r W_o$, we resort again to the Arnoldi algorithm

Balanced Truncation Via Arnoldi

- Procedure BTVA
  - Use Krylov-subspace method to calculate W_r, W_o
  - Use Arnoldi algorithm to calculate big eigenvalues and coresponding left and right eigenvectors $(V_{r,k}, V_{l,k})$ of $W_r W_o$
  - Choose the degree for reduced order system based on calculated eigenvalues and the desired error bounds
  - Compute $E = V_{L,k}{}^T V_{R,k}$ $\quad U_E \Sigma_k V_E{}^T = E$

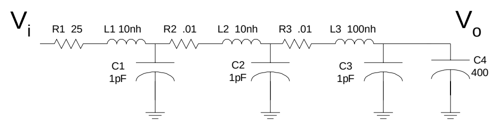
$$S_L = V_{L,k} U_E \Sigma^{-1/2} \in \Re^{n \times k} \qquad S_R = V_{R,k} V_E \Sigma^{-1/2} \in \Re^{n \times k}$$

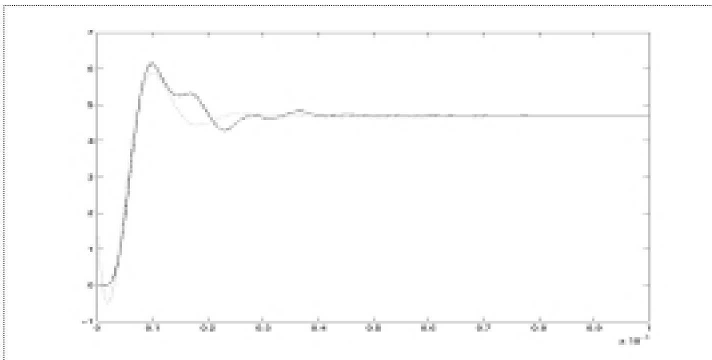
  - Compute the reduced order system

$$\hat{A} = S_L{}^T A S_R \qquad \hat{B} = S_L{}^T B \qquad \hat{C} = C S_R \qquad \hat{D} = D$$

Experimental Results

- Small Circuit:
- 8->4 identical
- 8->2 Some error





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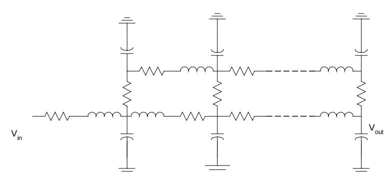
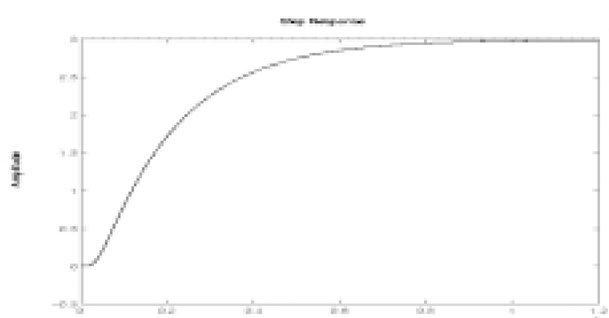
Experimental Result

- Big System
- 320-> 4 : Identical





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Conclusions

- Balanced realization is a provably optimal solution to order reduction of LTI systems
- It results in better reduced system compared to the Pade-based techniques
- The computational complexity may however limit the application of this method
- Better methods for solving Lyapunov equations are required to handle higher order systems